CURVED FINITE ELEMENTS AND CURVE APPROXIMATION

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The approximation of parameterized curves by segments of parabolas that pass through the endpoints of each curve segment arises naturally in all quadratic isoparametric transformations. While not as popular as cubics in curve design problems, the use of parabolas allows the introduction of a geometric measure of the discrepancy between given and approximating curves. The free parameters of the parabola may be used to optimize the fit, and constraints that prevent overspill and curve degeneracy are introduced. This leads to a constrained optimization problem in two variables that can be solved quickly and reliably by a simple method that takes advantage of the special structure of the problem.

FORMULATION OF THE PROBLEM

We assume that tessellation has been done in such a way that every curve segment lies to one side of the straight line segment that connects consecutive nodes, and that the approximating curve segment results from a quadratic isoparametric transformation and is therefore parabolic [1] and has the nodes as endpoints. The free parameters of the transformation are the coordinates of that point (α,β) on the approximating parabola where the slope is identical to that of the node-connecting straight-line segment. We impose constraints that prevent image overspill (equivalently, vanishing of the transformation Jacobian over an associated triangular master element) and ensure limit stability as the curve degenerates to a straight line. We introduce coordinate axes in such a way that the nodes are at $(-1,0)$ and $(1,0)$, with the curve segment above the horizontal x-axis. The approximating parabolic segments can then be expressed in parameterized form as

$$x = t + \alpha(1 - t^2), \; y = \beta(1 - t^2), \; -1 \le t \le 1 \tag{1}$$

where the parameters α and β completely determine a particular member of the approximating set (1). The axis of the parabola has slope $\frac{\beta}{\alpha}$. The constraints associated with overspill have the form

$$\tfrac{1}{2} - \alpha - \frac{1}{\mu_1}\beta > 0, \; \tfrac{1}{2} + \alpha - \frac{1}{\mu_2}\beta > 0 \tag{2}$$

where μ_1 and μ_2 are constants that are related to the transformation [2]. We ensure limit stability by restricting the axial slope [1], namely

$$-\frac{1}{m} \le \frac{\alpha}{\beta} \le \frac{1}{m} \tag{3}$$

where m is a positive stability safety factor. The constraints (2) and (3) define the feasible domain in which the point (α,β) may lie.

The parameters α and β may be used to optimize the approximation of a given curve

$$x = A(s),\ y = B(s),\ -1 \le s \le 1 \tag{4}$$

with

$$A(1) = -A(-1) = 1,\ B(1) = B(-1) = 0,\ B(s) > 0 \text{ for } -1 < s < 1$$

once we have defined a discrepancy measure for (1) and (4). When the approximating curves are parabolas, a uniquely defined distance between any point on the given curve and a parabola can be measured in the direction of the axis of the parabola. This results, in effect, in a re-parameterization of the parabola with respect to the given curve parameter s. Integration of the squares of the distances for the relevant parameter interval $-1 \le s \le 1$ yields the objective function

$$F(\alpha,\beta) = \left(\frac{\alpha^2}{\beta^2}+1\right) \int_{-1}^{1} (\beta(1-t^2(s)) - B(s))^2 ds \tag{5}$$

where

$$t(s) = A(s) - \frac{\alpha}{\beta} B(s)$$

This paper is concerned with the optimization of (5) subject to the constraints of (2) and (3).

THE OPTIMIZATION PROCEDURE

When the inverse $\frac{\alpha}{\beta}$ of the axial slope is considered as a new variable γ, (2), (3) and (5) can be expressed in terms of β and γ, and the equivalent constrained optimization problem can be formulated, namely, minimize

$$G(\beta,\gamma) = (\gamma^2+1)(\beta^2 C_2(\gamma) + \beta C_1(\gamma) + C_0) \tag{6}$$

where

$$C_2(\gamma) = \int_{-1}^{1} (1-t^2(s))^2 ds\ ,\ C_1(\gamma) = -2\int_{-1}^{1} B(s)(1-t^2(s))ds$$
$$C_0 = \int_{-1}^{1} B^2(s)ds\ ,\ t(s) = A(s) - \gamma B(s) \tag{7}$$

subject to

$$\beta \ge 0,\ -\frac{1}{m} \le \gamma \le \frac{1}{m},\ \tfrac{1}{2} - \beta\left(\gamma+\frac{1}{\mu_1}\right) > 0,\ \tfrac{1}{2} + \beta\left(\gamma-\frac{1}{\mu_2}\right) > 0 \tag{8}$$

The main advantage of the formulation (6) to (8) is that we can now use a separated optimization procedure [3] to find the best fit. We define the relevant objective function

$$\tilde{G}(\gamma) = \min_{0\le\beta\le\beta_{max}} G(\beta,\gamma) \tag{9}$$

where β_{max} is a function of γ, and β is restricted to the feasible domain by the upper limit. The function $\tilde{G}(\gamma)$ is optimized with respect to γ, for $-\frac{1}{m} \leq \gamma \leq \frac{1}{m}$, to find the optimal value $\bar{\gamma}$. The optimal $\bar{\beta}$ is obtained by introducing $\bar{\gamma}$ in (9) and optimizing with respect to β. Justification of this procedure is presented in [4], and its implementation is described in [5]. It is reliable, efficient and eminently suitable for use on smaller computers and in the absence of multi-purpose optimization software.

DISCUSSION OF APPLICATIONS

For applications in the field of computer-aided design, the given curves (4) are often cubic polynomials, and the coefficients (7) may be calculated in closed form in terms of the polynomial coefficients by using a symbolic machine language so that families of curves can be approximated with no further integration. For general curves, numerical quadrature may be used, as in the implementation [5] where the Romberg quadrature is applied. The coefficient functions $C_1(\gamma)$ and $C_2(\gamma)$ are expanded as polynomials in γ, so that for given A(s) and B(s) the integrations need only be done once.

The method was used to find optimal constrained parabolic approximation to a wide variety of given curves. Some examples from [6] were included in the numerical tests. A comprehensive discussion of the experimental results is contained in [4]. The method yielded satisfactory approximations to the given curves for all the examples considered.

REFERENCES

1. McLeod, R.J.Y.: Some applications of geometry in numerical analysis. Proceedings of the 9th Biennial Conference on Numerical Analysis, Dundee, 1981. Springer Verlag, Berlin, 1982.

2. Mitchell, A.R.; Wait, R.: The finite element method in partial differential equations. J. Wiley & Sons, London, 1977.

3. Golub, G.H.; Pereyra, V.: The differentiation of pseudo-inverses and nonlinear least squares problems whose variables separate. SIAM J. Numer. Anal. 10, 1973, pp. 413-432.

4. Baart, M.L.; McLeod, R.J.Y.: Constrained parabolic approximation of curves in the finite element method. NRIMS Technical Report TWISK 340, CSIR, Pretoria, December 1983.

5. Baart, M.L.: SPAC: A Fortran subroutine for constrained parabolic approximation of parametric curves. NRIMS Internal Report I521, CSIR, Pretoria, October 1983.

6. Mullineux, G.: Approximating shapes using parameterized curves. IMA. J. Appl. Math. 29, 1982, pp. 203-220.